Facets For Matroidal–Knapsack And Knapsack Problems

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Abstract

The Matroidal–Knapsack problem is the intersection of a Matroid with a Knapsack problems. Facets for small instances of Matroidal–Knapsack problems are derived from a family of cuts for the polyhedra of these problems. These facets can be lifted to facets of large problems by standard methods.

These facets could be used in a Branch and Cut Method, for this, we give some heuristics for the Separation problem which allow to obtain the cuts provided by this facets.

We also derive, using these cuts, new facets for Knapsack problems.

1 Introduction

Given a Matroid and a Knapsack System over the same ground set E, the *Matroidal–Knapsack System* is the intersection of both, that is, the independent sets are the subsets of E which are independent in both Matroid and Knapsack.

Examples of Matroidal–Knapsack problems are: (a) Knapsack with a constrained number of elements. (b) Multiple-choice Knapsack, the given elements are partitioned into disjoint sets, and the number of elements to be selected from each set is bounded from above. (c) Tree Knapsack, the elements are edges of a given graph, and the combinatorial constrains forbids the selection of edges forming cycles.

A large family of cuts was given for Matroidal–Knapsack Polyhedra in [1].

Here we give stronger results by showing how these cuts give facets of small problems or how to *twist* them to obtain facets. These facets can be lifted to facets of large problems by standard methods.

These cuts and facets could be combined with Matroid and Knapsack cuts in a Branch and Cut Method, for this we give some heuristics to obtain the new ones.

We also derive new facets for Knapsack Polyhedra.

2 Problem Statement And Definitions

Let $E = \{1, ..., n\}$ and $\mathcal{I} \subset 2^E$ a non-empty hereditary family of subsets of E, that is, such that $I \subset J \in \mathcal{I}$ implies $I \in \mathcal{I}$. We then say that $(E, \mathcal{I})$ is an *Independent System* and $\mathcal{I}$ is called the family of *independent* sets. For any Independent System, we denote: by $\mathcal{B}$ the maximal independent sets or *bases*, by $\mathcal{D} = \{D \subseteq E \mid D \notin \mathcal{I}\}$ the family of all subsets of E which are not independent or *dependent sets* and by $\mathcal{C}$ the family of all minimal dependent sets or *circuits* .

We shall be concerned about Independent Systems, one corresponding to the *Knapsack System*, where, given a positive integer b and a non-negative integer weight a_j for each element j of E, the family of independent sets, denoted by $\mathcal{K}$, is the family of subsets I of E such that $\sum_{j\in I} a_j \leq b$.

The other special Independent System we shall be considering is obtained when the following property holds for $\mathcal{I}$: for all I, $J \in \mathcal{I}$ such that $|I| = |J|+1$, there exists an element k of $I \setminus J$ such that $I \cup \{k\} \in \mathcal{I}$. In this case the Independent System is called a *Matroid* and its family of independent sets is denoted by $\mathcal{M}$.

This work studies properties of the Independent System resulting of the intersection of a Matroid and a Knapsack Systems defined over the same ground set E. We then speak of a *Matroid–Knapsack System* or problem and we denote by $\mathcal{L}$ the family of its independent sets. We analyze the polyhedral structure of these problem types.

The polyhedral approach identifies an Independent System using the characteristic vector x^I given for each I by $x^I_j = \{1 \text{ if } j \in I,\ 0 \text{ otherwise}\}$ and considering the convex hull of these vectors, i. e. $P_{\mathcal{I}} := conv\{x^I \in \{0,1\}^n | \ I \in \mathcal{I}\}$.

For any Independent System, $P_{\mathcal{I}}$ has the property that $0 \leq x \leq y \in P_{\mathcal{I}}$ implies $x \in P_{\mathcal{I}}$.

For any circuit C of an Independent System, a valid inequality for the corresponding polyhedron $P_{\mathcal{I}}$ is given by *Circuit Inequality*: $\sum_{j\in C} x_j \leq |C| - 1$.

We use $x(A)$ as a short-hand for $\sum_{j\in A} x_j$, for any subset A of E.

For any $A \subseteq E$, a valid inequality for $P_{\mathcal{I}}$ is given by the *Rank Inequality*: $x(A) \leq r_{\mathcal{I}}(A)$, where $r_{\mathcal{I}}(A) = \max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq A\}$, is called the *rank function* of the Independent System $(E, \mathcal{I})$.

It is well known that $P_{\mathcal{M}}$ has been fully characterized in [2] as the set of real vectors x satisfying the Rank Inequalities and non-negativity. As for as $P_{\mathcal{K}}$ is concerned, no full characterization is known, but this polyhedron has been extensively studied and several families of facet-inducing inequalities have been found. The reader is referred to [6] as a text in Combinatorial Optimization for polyhedral theory of combinatorial problems.

3 Valid Inequalities For $P_{\mathcal{I}}$

In this section we will present a general theorem about valid inequalities for the polyhedron $P_{\mathcal{I}}$ of any Independent System.

We began stating the results, proven in our work [1] in PANEL'92, that we need in this paper.

Notation 3.1 Trough the paper we denote: by (P, Q, S, T) a partition of E, with cardinalities p, q, s, t respectively; by k the rank of $(P \cup T)$ in the Matroid $(E, \mathcal{M})$; by q_0 a distinguish element of Q.

Assumption 3.2 We assume: (a) $t \geq 2$, $q \geq 1$, $0 \leq p < k < p+t$. (b) $r_{\mathcal{M}}(Q \cup P \cup T) = k+q$. (c) The set $P \cup T$ is dependent in the Matroid $(E, \mathcal{M})$. (d) For all $e \in T$ the set $P \cup Q \cup \{e\}$ is a circuit of the Knapsack System $(E, \mathcal{K})$. (e) The set S, which represent the elements not used in the configuration, could be empty.

The main result of [1] is the next theorem.

Theorem 3.3 The following inequalities are cuts for $P_{\mathcal{L}}$:

$$x(T) + tx(P) + (t-1)x(Q) \leq t(p+q) - q \tag{1}$$

$$x(T \cup Q) + 2x(P) \leq k + p + q - 1 \tag{2}$$

When t and $k + t(p+q) + q$ are odd:

$$x(T) + \frac{t+1}{2}x(P \cup Q) \leq \frac{k + t(p+q) + q - 1}{2} \tag{3}$$

When t and $k + t(p+q) + p$ are even:

$$x(T) + \frac{t+2}{2}x(P) + \frac{t}{2}x(Q) \leq \frac{k + t(p+q) + p - 2}{2} \tag{4}$$

□

The proof of Theorem 3.3 depends only on the fact that the following inequalities are valid for $P_{\mathcal{L}}$.

$$0 \leq x_e \leq 1 \; \forall e \in E \tag{5}$$

$$x(P \cup T) \leq k \tag{6}$$

$$x(P \cup Q \cup \{e\}) \leq p + q \; \forall e \in T \tag{7}$$

The inequalities in (5) are valid for any Independent System. The inequalities in (6) and (7) are a direct consequence of the Rank Inequality and Assumption 3.2.

The inequalities in (5), (6) and (7) imply that the inequalities in Theorem 3.3 are valid and they cut the fractional vertex x^o defined as:

$$x^o_e = \begin{cases} 1 & \text{if } e \in P \cup Q \setminus \{q_0\} \\ \frac{k-p}{t} & \text{if } e \in T \\ 1 - \frac{k-p}{t} & \text{if } e = q_0 \\ 0 & \text{if } e \in S \end{cases} \tag{8}$$

As a direct consequence of this we have:

Theorem 3.4 For any Independent System such that the inequalities in (6) and (7) are valid for $P_{\mathcal{I}}$ we have that the inequalities in Theorem 3.3 are valid. □

4 Facets For $P_{\mathcal{L}}$ And $P_{\mathcal{K}}$

In this section we shows how to derive conditions which assure us that some of the inequalities in Theorem 3.3 correspond to facets of $P'_{\mathcal{L}} = P_{\mathcal{L}} \cap \{x_e = 0 \mid e \in S\}$.

Hence, we can use Lifting Techniques to obtain facets of $P_{\mathcal{L}}$. These techniques are presented in [7] for Knapsack Problems.

To show that a given valid inequality of a polyhedron P is facet inducing it is enough to show that that there are dimension of P points in P which are linear independent and satisfy the given inequality as equality. Notice that dimension of $P'_{\mathcal{L}}$ is $p+q+t$.

Polyhedron 4.1 Let $E = \{1, \ldots, t, p_0, q_0\}$, $T = \{1, \ldots, t\}$, $P = \{p_0\}$ and $Q = \{q_0\}$, with $p = q = 1$ and $k = t$ and let $T \cup Q \in \mathcal{K}$. In this case the dimension of $P'_{\mathcal{L}}$ is $t+2$ and $P \cup T$ is a circuit.

Lemma 4.2 The $t+2$ points: $x^{P\cup T\setminus\{e\}}, \forall e \in T, x^{P\cup Q}$, and$x^{T\cup Q}$, are linear independent and belong to $P'_{\mathcal{L}}$.

Proof: These points belong to $P'_{\mathcal{L}}$: since $P \cup T$ is a circuit, the sets $P \cup T \setminus \{e\} \in \mathcal{L}$ and also $P \cup Q$ and $T \cup Q$ are in $\mathcal{L}$.

These points are linear independent since it is easy to check that the associated inverse matrix is:

$$\begin{pmatrix} 0 & 1 & \dots & 1 & 1 & 0 \\ 1 & 0 & \dots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2t-1} \begin{pmatrix} 3-2t & 2 & \dots & 2 & -1 & 1 \\ 2 & 3-2t & \dots & 2 & -1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 2 & \dots & 3-2t & -1 & 1 \\ 1 & 1 & \dots & 1 & t-1 & 1-t \\ -1 & -1 & \dots & -1 & t & t-1 \end{pmatrix}$$

□

Theorem 4.3 The inequality (1) is facet inducing for $P'_{\mathcal{L}}$ defined in Polyhedron 4.1.

Proof: In this case the inequality (1) is: $x(T) + tx(P) + (t-1)x(Q) \leq 2t - 1$ which is satisfied as equality by the points in Lemma 4.2. □

Not all the inequalities give facets, for example, the inequality (2) in this case is $x(T \cup Q) + 2x(P) \leq t + 1$ it is, in general, not facet inducing for $P'_{\mathcal{L}}$ in Polyhedron 4.1, since it is satisfied as equality for all the points in (4.2) but $x^{T \cup Q}$ (with the exception of the case $t = 2$), but the points in (4.2) correspond to the bases of $\mathcal{L}$.

However, it is easy to derive facets for other configurations, we close with two cases where inequality (2) and (4) are facet inducing for $P'_{\mathcal{L}}$.

Example 4.4 Let $E = \{1, \ldots, 6\}$, $T = \{1, 2\}$, $P = \{3, 4\}$ and $Q = \{5, 6\}$, with $k = 3$. In this case the dimension of $P'_{\mathcal{L}}$ is 6. They correspond to a Matroid with one circuit $\{1, 2, 3, 4\}$, and a Knapsack System with equation $x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + 2x_6 \leq 8$.

It is easy to check that the following six point corresponding to the rows of the matrix are independent in both systems, satisfied (2) as equality (in this case is $x(T \cup Q) + 2x(P) \leq 6$) and they are linear independent:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 1 & -3 & -3 & 1 & 1 & 4 \\ 1 & 3 & -3 & 1 & 1 & -2 \\ 2 & 0 & 0 & 2 & -4 & 2 \\ -4 & 0 & 0 & 2 & 2 & 2 \\ 1 & -3 & 3 & 1 & 1 & -2 \\ 1 & 3 & 3 & -5 & 1 & -2 \end{pmatrix}$$

Polyhedron 4.5 Let $E = \{1, \ldots, t, p_0, q_0\}$, $T = \{1, \ldots, t\}$, $P = \{p_0\}$ and $Q = \{q_0\}$, with $p = q = 1$ and $k = t - 1$. We assume that $\{1, 2\}$ is a circuit in the Matroid and that $T \setminus \{1\} \cup Q \in \mathcal{K}$. In this case the dimension of $P'_{\mathcal{L}}$ is $t + 2$.

Lemma 4.6 The $t+2$ points in $P'_{\mathcal{L}}$ of Polyhedron 4.5:

$$x^{P\cup T\setminus\{2,3\}};\ x^{P\cup T\setminus\{1,e\}}\ \forall e \in T,\ \ e \neq 1;\ x^{P\cup Q};\ x^{T\setminus\{1\}\cup Q}$$

are linear independent and belong to $P'_{\mathcal{L}}$.

Proof: These points belong to $P'_{\mathcal{L}}$, the sets $P \cup T \setminus \{e\} \in \mathcal{L}$ because $P \cup T$ is a circuit and also $P \cup Q$ and $T \cup Q$ are in $\mathcal{L}$.

These points are linear independent since it is easy to check that the associated inverse matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \dots & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}^{-1} =$$

$$\frac{1}{2t-3}\begin{pmatrix} 2t-3 & 5-2t & 5-2t & 2 & \dots & 2 & -1 & 1 \\ 0 & 5-2t & 2 & 2 & \dots & 2 & -1 & 1 \\ 0 & 2 & 5-2t & 2 & \dots & 2 & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 2 & 2 & \dots & 5-2t & -1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & t-2 & 2-t \\ 0 & -1 & -1 & -1 & \dots & -1 & t-1 & t-2 \end{pmatrix}$$

□

Lemma 4.7 The inequality (4) is facet inducing for $P'_{\mathcal{L}}$ in Polyhedron 4.5 when $t=4$.

Proof: We can apply (4) since $k+t(p+q)+p = 3+8+1 = 12$ is even. In this case the inequality (4) is: $x(T) + 3x(P) + 2x(Q) \leq 5$ which is satisfied as equality by the points in (4.6). □

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 1 & -3 & -3 & 1 & 1 & 4 \\ 1 & 3 & -3 & 1 & 1 & -2 \\ 2 & 0 & 0 & 2 & -4 & 2 \\ -4 & 0 & 0 & 2 & 2 & 2 \\ 1 & -3 & 3 & 1 & 1 & -2 \\ 1 & 3 & 3 & -5 & 1 & -2 \end{pmatrix}$$

The inequality (4) is, in general, not facet inducing for $P'_{\mathcal{L}}$ in Polyhedron 4.5, since it is satisfied as equality for all the points in (4.2) but $x^{T \cup Q}$, with the exception of $t = 4$. We will use this inequality to show how to *twist* it to include this point.

Theorem 4.8 When t is even, the inequality $x(T)+(t-1)x_{p_0}+(t-2)x_{q_0} \leq 2t-3$ obtained by twisting (4) to include $x^{T \cup Q}$, is facet inducing for $P'_{\mathcal{L}}$ in Polyhedron 4.5.

Proof: Let $t = 2\ell$. We can apply (4) since $k + t(p+q) + p = 2\ell - 1 + 4\ell + 1 = 6\ell$. In this case the inequality (4) is: $x(T) + (\ell+1)x(P) + \ell x(Q) \leq 3\ell - 1$ which is satisfied as equality by the points in (4.6) but $x^{T \cup Q}$. To include this point we need the inequality

$$\alpha_1 x(T) + \alpha_2 x_{p_0} + \alpha_3 x_{q_0} \leq 3\ell - 1 \tag{9}$$

satisfying:

$$\begin{cases} (2\ell-2)\alpha_1 + \alpha_2 = 3\ell - 1 \\ +\alpha_2 + \alpha_3 = 3\ell - 1 \\ (2\ell-1)\alpha_1 + \alpha_3 = 3\ell - 1 \end{cases}$$

Whose solution is $\alpha_1 = \frac{3\ell-1}{4\ell-3}$, $\alpha_2 = (2\ell-1)\alpha_1$ and $\alpha_3 = (2\ell-2)\alpha_1$. With these values in (9) and dividing by α_1, we obtain

$$x(T) + (2\ell-1)x_{p_0} + (2\ell-2)x_{q_0} \leq 4\ell - 3,$$

with $2\ell = t$ we obtain the inequality of the theorem. □

Is easy to check that the inequality in Theorem 4.8 is equal to the inequality in Lemma 4.7 when $t = 4$.

Since we used only the matrix of circuit configurations, given by the inequalities (6) and (7), to derive valid inequalities, we could use the circuit inequalities of the Knapsack to obtain facets of it. The next case shows how it could be done.

Polyhedron 4.9 Let $E = \{1, \ldots, t, t+1\}$, $T = \{1, \ldots, t\}$, $P = \emptyset$ and $Q = \{t+1\}$, with $p = 0$, $q = 1$ and $k = t - 1$, corresponding to the Knapsack inequality

$$\sum_{j=1}^{t-1} 2x_j + 4x_t + 2tx_{t+1} \leq 2t + 1.$$

In this case the dimension of P'_K is $t + 1$.

Lemma 4.10 The $t + 1$ points in P'_K of Polyhedron 4.9: $x^{T \setminus \{e\}} \forall e \in T$ and x^Q are linear independent.

Proof: Is easy to check that these points belong to P'_K. These points are linear independent since it is easy to check that the associated inverse matrix is:

$$\begin{pmatrix} 0 & 1 & \ldots & 1 & 1 & 0 \\ 1 & 0 & \ldots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{t-1} \begin{pmatrix} 2-t & 1 & \ldots & 1 & 1 & 0 \\ 1 & 2-t & \ldots & 1 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 2-t & 1 & 0 \\ 1 & 1 & \ldots & 1 & 2-t & 0 \\ 0 & 0 & \ldots & 0 & 0 & t-1 \end{pmatrix}$$

□

Theorem 4.11 The inequality (1) is facet inducing for P'_K in Polyhedron 4.9.

Proof: In this case the inequality (1) is: $x(T) + (t-1)x(Q) \leq t - 1$ which is satisfied as equality by the points in Lemma 4.10. □

These facets are new ones since they are neither facets from a 1-configuration [8] nor from a cover [6].

5 Separation Heuristics For $P_{\mathcal{L}}$

In this section we consider some examples of separation heuristics for $P_{\mathcal{L}}$ than can be used in a Branch and Cut Method. In [4] and [5] there are descriptions of this method and a general presentation in [6].

Heuristic 5.1 Given a solution x to the linear relaxation, by (8) we are looking for a dependent set in the Matroid with values α and 1 in x, the 1's correspond to P and the others to T. Then look for an arc, independent from the circuit, with value $1 - \alpha$ corresponding to q_0 and check if one of the corresponding inequalities is violated, if not, try to add elements to P and Q.

5.2 Similarly to the graph defined in [3] and the technique used in [4], we define the Submatroid M_α in the ground set $A = \{a \in E |\ x(a) = 1 \vee x(a) = \alpha \pm \epsilon\}$, $x(a)$ is the weight of the element a.

Heuristic 5.3 For each α present we could look in M_α for small configurations, which we could enumerate, like presented in [4], for Three Fences.

Heuristic 5.4 For each α present we could look in M_α for a maximum base, this gives a circuits basis and we have a configuration for each element not in the basis.

Heuristic 5.5 For any Matroid, we could define the 1-forest Matroid, where we admit at most one circuit in an independent set. For each α present we could look in M_α for a maximum 1-forest.

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